DOI: 10.1515/auom-2017-0011
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 25(1),2017, 131–143

On continuous dependence for the mixed problem of microstretch bodies

M. Marin, I. Abbas and C. Cârstea

Abstract

We do a qualitative study on the mixed initial-boundary value problem in the elastodynamic theory of microstretch bodies. After we transform this problem in a temporally evolutionary equation on a Hilbert space, we will use some results from the theory of semigroups of linear operators in order to prove the continuous dependence of the solutions upon initial data and supply terms.

1. Introduction

There are many studies on generalized structures of continuum bodies. One of the pioneers of these theories was Eringen and among his many studies on these theories, we emphasize only the papers [5] and [6]. As it is known, in order to describe adequately the behaviour of materials such a liquid crystal, fluid suspensions, polycrystalline aggregates and granular media, it is necessary to introduce into the continuum theory some terms reflecting the microstructure of the materials. This microstructure includes intrinsic rotations and microstructural expansion and contractions. In the context of this theory, each material point has three deformable directors. We remember that a continuum body is a microstretch continuum if the directors are constrained to have only breathing-type microdeformations. All point of a microstretch continuum can stretch and contract independently of their translations and

Key Words: Elasticity; Microstretch; Evolution equations; Semi-group of contractions.
2010 Mathematics Subject Classification: Primary 74A15; Secondary 47A45, 35D30.
Received: 11.02.2016.
Accepted: 29.04.2016.

rotations. Already this theory has found applications in the treatment of composites materials reinforced with chopped fibers. Also, this theory is useful in the applications which deal with porous materials as geological materials, solid packed granular materials and many others. In the last decades have been published many papers which are concerned with this theory, of which we mention only some of the most recent. So, in [3] some of the basic results deduced by Eringen are used in order to investigate the isothermal bending of microstretch elastic plates. The paper [4] is a study of some basic properties of wave numbers of the longitudinal and transverse plane harmonic waves, in the context of thermoelasticity for materials with voids. Iesan and Pompei in [8] and Iesan and Nappa in [9] have approached some problems of microstretch elastic materials. In our studies [12]-[16] we tackle some questions with regards to the microstretch thermoelastic materials. Some results on vibrations can be found in [18].
Our present paper must be considered as a good tool for a better understanding of microstretch in the study of above enumerated materials.

2. Basic equations

In the following we will consider an anisotropic and homogeneous body which occupies, at time $t = 0$, a properly regular domain B of the three-dimensional Euclidian space R^3. The domain B is bounded by the piece-wise smooth surface ∂B. If we denote by $\bar{B}$ the closure of B, then we have $\bar{B} = B \cup \partial B$. As usual, we will report the motion of the body to a fix system of rectangular Cartesian axes $Ox_i, i = 1, 2, 3$ and we will adopt the Cartesian tensor notation. The notations x_j are for spatial coordinates of points in B and $t \in [0, \infty)$ is temporal variable. Also, we will use the Einstein summation convention over repeated indices. is used. By convention, the subscript j after comma indicates partial differentiation with respect to the spatial argument x_j and a superposed dot denotes the derivatives with respect to the $t-$ time variable. All Greek indices have the range $(1, 2)$, while all Latin subscripts are understood to range over the integers $(1, 2, 3)$. The spatial argument and the time argument of a function will be omitted, when there is no likelihood of confusion.

The behavior of a microstretch body will be described by using the variables (u_i, φ_i, ϕ) where we denote by u_i the components of the displacement vector, by φ_i the components of the microrotation vector and denote by ϕ a scalar function that characterizes the microstretch.

As usual, we will denote the components of the stress by $(t_{ij}, m_{ij}, \lambda_i)$, where t_{ij} are the components of the stress tensor over B, m_{ij} are the components of the couple stress tensor over B and λ_i are the components of the microstress vector.

We will restrict our considerations to the case of an elastic microstretch body which is free of initial stress and couple stress and has zero intrinsic equilibrated body forces. In this case the internal energy density has the form

$$\varrho_0 e = \frac{1}{2} A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + B_{ijmn} \varepsilon_{ij} \mu_{mn} + \frac{1}{2} C_{ijmn} \mu_{ij} \mu_{mn} + \\ + \frac{1}{2} F_{ij} \gamma_i \gamma_j + D_{ijk} \varepsilon_{ij} \gamma_k + E_{ijk} \mu_{ij} \gamma_k, \tag{1}$$

With the help of the internal energy density we can deduce the fundamental system of field equations of the mixed problem in the dynamic theory of Elasticity of microstretch bodies. This consists of

- the equations of motion

$$t_{ji,\,j} + \varrho_0 F_i = \varrho_0 \ddot{u}_i, \\ m_{ji,\,j} + \varepsilon_{ijk} t_{jk} + \varrho_0 G_i = I_{ij} \ddot{\varphi}_j \tag{2}$$

- the balance of the equilibrated forces

$$\lambda_{i,\,i} + \varrho_0 L = \varrho_0 \kappa \ddot{\phi}. \tag{3}$$

The constitutive equations, specific for an anisotropic and homogeneous microstretch thermoelastic material, have the form

$$t_{ij} = A_{ijmn}\, \varepsilon_{mn} + B_{ijmn} \mu_{mn} + D_{ijk} \gamma_k, \\ m_{ij} = B_{ijmn}\, \varepsilon_{mn} + C_{ijmn} \mu_{mn} + E_{ijk} \gamma_k, \\ \lambda_i = D_{mni}\, \varepsilon_{mn} + E_{mni} \mu_{mn} + F_{ij} \gamma_j. \tag{4}$$

The constitutive coefficients A_{ijmn}, B_{ijmn}, C_{ijmn}, D_{ijk}, E_{ijk}, F_{ij}, a_{ij}, b_{ij}, c_i, d and k_{ij} characterize the properties of materials and satisfy the following symmetry relations

$$A_{ijmn} = A_{mnij},\ C_{ijmn} = C_{mnij},\ F_{ij} = F_{ji},\ k_{ij} = k_{ji}. \tag{5}$$

In the above equations we have used the following notations:

- F_i the components of body force;
- G_i the components of body couple;
- L the generalized external body load;
- ϱ_0 is the reference constant mass density;
- J and $I_{ij} = I_{ji}$ are the coefficients of microinertia;
- κ the equilibrated inertia.

In papers [5] and [6] can be found more detailed physical significances of the

functions L, λ_i and κ.
To characterize the deformation we used notations $(\varepsilon_{ij}, \mu_{ij}, \gamma_i)$ where the strain tensors are defined by means of the following geometric equations

$$\varepsilon_{ij} = u_{j,\,i} + \varepsilon_{ijk}\varphi_k,\ \mu_{ij=}\varphi_{j,\,i},\ \gamma_i = \phi_{,\,i}, \tag{6}$$

where ε_{ijk} is the alternating symbol.
To characterize the surface traction at regular points of the surface ∂B, we will use the quantities (t_i, m_i, λ) where t_i are the components of surface traction, m_i are the components of surface couple and λ is the microsurface traction and are defined by

$$t_i = t_{ji}n_j,\ m_i = m_{ji}n_j,\ \lambda = \lambda_i n_i,\ q = q_i n_i.$$

As usual, we denoted by n_i the components of the outward unit normal of the surface ∂B.
We'll complete problem the mixed initial boundary value problem for elasticity of microstretch bodies by adding to the system of field equations (1)-(4) of the boundary conditions

$$u_i(x,t) = 0,\ \varphi_i(x,t) = 0,\ \phi(x,t) = 0,\ (x,t) \in \partial B \times [0, t_0]\,, \tag{7}$$

and the initial conditions:

$$\begin{aligned} &u_i(x,0) = a_i(x),\ \ \dot{u}_i(x,0) = b_i(x),\\ &\varphi_i(x,0) = c_i(x),\ \ \dot{\varphi}_i(x,0) = d_i(x),\ x \in \bar{B}\\ &\phi(x,0) = \phi^0(x),\ \ \dot{\phi}(x,0) = \phi^1(x) \end{aligned} \tag{8}$$

The significance of the given functions a_i, b_i, c_i, d_i, ϕ^0 and ϕ^1 is obvious.
If we substitute the constitutive equations (4) and the geometric equations (6) into equations (2) and (3), we obtain in the following system of coupled equations:

$$\begin{aligned} \varrho_0\ddot{u}_i &= \left[A_{ijmn}\left(u_{n,\,m} + \varepsilon_{nmk}\varphi_k\right) + B_{ijmn}\varphi_{n,\,m} + D_{ijk}\phi_{,\,k}\right]_{,\,j} + \varrho_0 F_i,\\ I_{ij}\ddot{\varphi}_j &= \left[B_{ijmn}\left(u_{n,\,m} + \varepsilon_{nmk}\varphi_k\right) + C_{ijmn}\varphi_{n,\,m} + E_{ijk}\phi_{,\,k}\right]_{,\,j} +\\ &\quad + \varepsilon_{ijk}\left[A_{jkmn}\left(u_{n,\,m} + \varepsilon_{nms}\varphi_s\right) + B_{jkmn}\varphi_{n,\,m} + D_{jkr}\phi_{,\,r}\right] + \varrho_0 G_i,\\ \varrho_0\kappa\ddot{\sigma} &= \left[D_{mni}\left(u_{n,\,m} + \varepsilon_{nms}\varphi_s\right) + E_{mni}\varphi_{n,\,m} + A_{ij}\phi_{,\,j}\right]_{,\,i} + \varrho_0 L \end{aligned} \tag{9}$$

By summarizing, a solution of the mixed initial boundary value problem in the theory of elasticity of microstretch bodies in the cylinder $\Omega_0 = B \times [0, t_0]$ is an ordered array $(u_i,\ \varphi_i,\ \phi)$ which satisfies the system of equations (9) for all $(x,t) \in \Omega_0$, the initial conditions (8) and the boundary conditions (7).

3. Main results

In order to obtain the next results of this section, we need some assumptions imposed to quantities used in previous relations. These assumptions of positivity are in agreement with the usual restrictions imposed in the Mechanics of solids in order to obtain some basic results, such as the existence, the uniqueness or the continuous dependence of solution. So, for all arbitrary $\xi_{ij},\ \eta_{ij},\ \kappa_i$ we have

i) $\varrho_0 > 0,\ I_{ij} > 0,\ \kappa > 0,$

ii) $$A_{ijmn}\xi_{ij}\xi_{mn} + 2B_{ijmn}\xi_{ij}\eta_{mn} + C_{ijmn}\eta_{ij}\eta_{mn} + \\ +2D_{ijs}\xi_{ij}\kappa_s + 2E_{ijs}\eta_{ij}\kappa_s + A_{ij}\kappa_i\kappa_j \geq \\ \geq \alpha_0\left(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij} + \kappa_i\kappa_i\right),\ \alpha_0 > 0,$$

Due to the mathematical apparatus provided by the semigroups of linear operators, we will see that we can derive the above mentioned results without recourse to supplementary restrictions.
Let us define the space X by:

$$X = \left\{(\mathbf{u},\ \mathbf{v},\ \boldsymbol{\varphi},\ \boldsymbol{\psi},\ \phi,\ \alpha)\ /\ \mathbf{u} \in \mathbf{H}_0^1(B),\ \mathbf{v} \in \mathbf{H}^0(B),\right. \\ \left.\boldsymbol{\varphi} \in \mathbf{H}_0^1(B),\ \boldsymbol{\psi} \in \mathbf{H}^0(B),\ \phi \in H_0^1(B),\ \alpha \in H^0(B)\right\}. \quad (10)$$

Here we used the vectorial notations

$$\mathbf{u} = (u_i)\,,\ \mathbf{v} = (v_i)\,,\ \boldsymbol{\varphi} = (\varphi_i)\,,\ \boldsymbol{\psi} = (\psi_i)\,,\ i = 1, 2, 3. \quad (11)$$

and $H_0^1(B)$ and $H^0(B)$ are the familiar Sobolev spaces (see [1]) and we have

$$\mathbf{H}_0^1(B) = \left[H_0^1(B)\right]^3,\ \mathbf{H}^0(B) = \left[H^0(B)\right]^3. \quad (12)$$

By using an accessible procedure the space X can be equipped with a structure of Hilbert space. The mixed initial-boundary value problem, given by the equations (9), the initial conditions (8) and the boundary conditions (7) will be transformed into a temporally evolutionary equation in the Hilbert space

X defined in (10). To this aim we will need the operators

$$\begin{aligned}
&A_i\,\boldsymbol{\omega}=v_i,\\
&B_i\,\boldsymbol{\omega}=\frac{1}{\varrho_0}\left[A_{ijmn}\left(u_{n,\,m}+\varepsilon_{nmk}\varphi_k\right)+B_{ijmn}\varphi_{n,\,m}+D_{ijk}\phi_{,\,k}\right]_{,\,j},\\
&C_i\,\boldsymbol{\omega}=\psi_i,\\
&D_i\,\boldsymbol{\omega}=\frac{1}{I_{ij}}\left[B_{ijmn}\left(u_{n,\,m}+\varepsilon_{nmk}\varphi_k\right)+C_{ijmn}\varphi_{n,\,m}+E_{ijk}\phi_{,\,k}\right]_{,\,j}+\\
&\qquad+\varepsilon_{nmk}\left[A_{jkmn}\left(u_{n,\,m}+\varepsilon_{nms}\varphi_s\right)+B_{jkmn}\varphi_{n,\,m}+D_{jkr}\phi_{,\,r}\right],\\
&E\,\boldsymbol{\omega}=\alpha,\\
&F\,\boldsymbol{\omega}=\frac{1}{\varrho_0\kappa}\left[E_{mni}\left(u_{n,\,m}+\varepsilon_{nms}\varphi_s\right)+E_{mni}\varphi_{n,\,m}+A_{ij}\phi_{,\,j}\right]_{,\,i}
\end{aligned}\tag{13}$$

Now we build the matrix operator $\mathcal{L}$ by

$$\mathcal{L}\,\boldsymbol{\omega}=(\mathbf{A}\,\boldsymbol{\omega},\ \mathbf{B}\,\boldsymbol{\omega},\ \mathbf{C}\,\boldsymbol{\omega},\ \mathbf{D}\,\boldsymbol{\omega},\ E\,\boldsymbol{\omega},\ F\,\boldsymbol{\omega}),\tag{14}$$

where we used the vector notations

$$\mathbf{A}=(A_i)\,,\ \mathbf{B}=(B_i)\,,\ \mathbf{C}=(C_i)\,,\ \mathbf{D}=(D_i)\,,\ i=1,\ 2,\ 3$$

We consider the domain of the operator $\mathcal{L}$ as

$$D=D(\mathcal{L})=\{\,\boldsymbol{\omega}\in X\ /\ \mathcal{L}\,\boldsymbol{\omega}\in X,\ \mathbf{v}=0,\ \boldsymbol{\psi}=0,\ \alpha=0\ \ on\ \ \partial B\}\,.\tag{15}$$

We can see that the domain $D(\mathcal{L})$ is not empty because it contains at least the space $\left[C_0^\infty(B)\right]^7$. Furthermore, the domain $D(\mathcal{L})$ is a set dense in X because the closure of $D(\mathcal{L})$ is the space X.
After all these considerations, we transform the initial-boundary value problem consists of equations (9) and conditions (7) and (8) to the temporally equation on the Hilbert space X

$$\frac{d\,\boldsymbol{\omega}}{dt}=\mathcal{L}\,\boldsymbol{\omega}+\mathcal{F}(t),\ 0\le t\le t_0,\tag{16}$$

with the initial condition

$$\boldsymbol{\omega}(0)=\boldsymbol{\omega}_0,\tag{17}$$

where we used the notations

$$\begin{aligned}
&\mathcal{F}(t)=\left(\mathbf{0},\ \mathbf{F},\ \mathbf{0},\ \mathbf{G},\ \mathbf{0},\ \frac{1}{\kappa}L\right),\ \boldsymbol{\omega}_0=\left(\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d},\ \phi^0,\ \phi^1\right),\\
&\mathbf{F}=(\varrho_0F_i)\,,\ \mathbf{G}=(\varrho_0G_i)\,,\ \mathbf{a}=(a_i)\,,\ \mathbf{b}=(b_i)\,,\ \mathbf{c}=(c_i)\,,\ \mathbf{d}=(d_i)\,.
\end{aligned}$$

The inner product $\langle \boldsymbol{\omega},\ \bar{\boldsymbol{\omega}} \rangle_*$ define by

$$\langle \boldsymbol{\omega},\ \bar{\boldsymbol{\omega}} \rangle_* = \int_B \left[\varrho_0 v_i \bar{v}_i + I_{ij} \psi_i \bar{\psi}_j + \varrho_0 \kappa \alpha \bar{\alpha} + A_{ijmn} \varepsilon_{ij} \bar{\varepsilon}_{mn} + \right. \\ + C_{ijmn} \mu_{ij} \bar{\mu}_{mn} + B_{ijmn} \left(\varepsilon_{ij} \bar{\mu}_{mn} + \bar{\varepsilon}_{ij} \mu_{mn} \right) + E_{ijk} \left(\varepsilon_{ij} \bar{\gamma}_k + \bar{\varepsilon}_{ij} \gamma_k \right) + \\ \left. + D_{ijk} \left(\mu_{ij} \bar{\gamma}_k + \bar{\mu}_{ij} \gamma_k \right) + A_{ij} \gamma_i \bar{\gamma}_j \right] dV, \tag{18}$$

can induce a norm such that we have new Hilbert space X^* equipped with this norm.
If we take into account the hypotheses $i)$ and $ii)$, the following inequality is obtained

$$|\boldsymbol{\omega}|_*^2 = \langle \boldsymbol{\omega},\ \boldsymbol{\omega} \rangle_* = \int_B \left[\varrho_0 v_i v_i + I_{ij} \psi_i \psi_j + \varrho_0 \kappa \alpha^2 \right] dV + \\ + \int_B \left[A_{ijmn} \varepsilon_{ij} \varepsilon_{mn} + 2 B_{ijmn} \varepsilon_{ij} \mu_{mn} + C_{ijmn} \mu_{ij} \mu_{mn} + \right. \\ \left. + 2 D_{ijk} \varepsilon_{ij} \gamma_k + 2 E_{ijk} \mu_{ij} \gamma_k + A_{ij} \gamma_i \gamma_j \right] dV \geq \\ \geq \int_B \left[\varrho_0 v_i v_i + I_{ij} \psi_i \psi_j + \varrho_0 \kappa \alpha^2 \right] dV + \\ + \int_B \alpha_0 \left(\varepsilon_{ij} \varepsilon_{ij} + \mu_{ij} \mu_{ij} + \gamma_i \gamma_i \right) dV \geq c_1 \left| \boldsymbol{\omega} \right|_X^2 . \tag{19}$$

Also, from (18) by using first Korn inequality (see [7]), we deduce the inequality

$$|\boldsymbol{\omega}|_*^2 \leq c_2 \left| \boldsymbol{\omega} \right|_X^2 . \tag{20}$$

and this inequality (20) with inequality (19) give

$$c_1 \left| \boldsymbol{\omega} \right|_X^2 \leq |\boldsymbol{\omega}|_*^2 \leq c_2 \left| \boldsymbol{\omega} \right|_X^2$$

and this ensures the fact that the norm $|.|_*$ is equivalent to the original norm on the Hilbert space X.
The result in next theorem is an important property of the operator $\mathcal{L}$.

Theorem 1. *The operator $\mathcal{L}$ is dissipative.*

Proof. According to the definition of a dissipative operator, we have to prove the inequality

$$\langle \mathcal{L}\ \boldsymbol{\omega},\ \boldsymbol{\omega} \rangle_* \leq 0, \quad for\ \ all\ \ \boldsymbol{\omega} \in D(\mathcal{L}).$$

Based on definitions (11), we deduce

$$\langle \mathcal{L}\,\boldsymbol{\omega},\,\boldsymbol{\omega}\rangle_* = \int_B \Big\{ v_i \left[A_{ijmn}\left(u_{n,\,m} + \varepsilon_{nmk}\varphi_k\right) + B_{ijmn}\varphi_{n,\,m} + D_{ijk}\phi_{,\,k}\right]_{,\,j} +$$
$$+\psi_i \left[B_{ijmn}\left(u_{n,\,m} + \varepsilon_{nmk}\varphi_k\right) + C_{ijmn}\varphi_{n,\,m} + E_{ijk}\phi_{,\,k}\right]_{,\,j} +$$
$$+\psi_i\varepsilon_{ijk}\left[A_{jkmn}\left(u_{n,\,m} + \varepsilon_{nms}\varphi_s\right) + B_{jkmn}\varphi_{n,\,m} + D_{jkr}\phi_{,\,r}\right] +$$
$$+\alpha\left[D_{mni}\left(u_{n,\,m} + \varepsilon_{nms}\varphi_s\right) + E_{mni}\varphi_{n,\,m} + A_{ij}\phi_{,\,j}\right]_{,\,i} -$$
$$+A_{ijmn}\left(u_{n,\,m} + \varepsilon_{nmk}\varphi_k\right)\left(v_{j,\,i} + \varepsilon_{jis}\psi_s\right) + C_{ijmn}\psi_{n,\,m}\varphi_{j,\,i} +$$
$$+B_{ijmn}\left[\left(u_{j,\,i} + \varepsilon_{jis}\varphi_s\right)\psi_{n,\,m} + \left(v_{n,\,m} + \varepsilon_{nms}\psi_s\right)\varphi_{j,\,i}\right] + A_{ij}\phi_{,\,i}\alpha_{,\,j}$$
$$+D_{ijk}\left[\left(u_{j,\,i}+\varepsilon_{jis}\varphi_s\right)\alpha_{,\,k}+\left(v_{j,\,i}+\varepsilon_{jis}\psi_s\right)\phi_{,\,k}\right]+E_{ijk}\left(\varphi_{j,\,i}\alpha_{,\,k}+\psi_{j,i}\phi_{,\,k}\right)\Big\}dV$$

In this last relation we now use the Green-Gauss formula and the boundary condition (7), so that we obtain the desired result. ■
Another property of the operator $\mathcal{L}$ is given in the following theorem.

Theorem 2. *The operator $\mathcal{L}$ satisfies the range condition, that is:*

$$R(\lambda I - \mathcal{L}) = X, \;\; \lambda > 0. \tag{21}$$

Proof. If we use the notation $\tilde{\boldsymbol{\omega}} = \left(\tilde{\mathbf{u}},\,\tilde{\mathbf{v}},\,\tilde{\boldsymbol{\varphi}},\,\tilde{\boldsymbol{\psi}},\,\tilde{\phi},\,\tilde{\alpha}\right) \in X$, then the range condition means that we have to show that the equation

$$\lambda\boldsymbol{\omega} - \mathcal{L}\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}}, \tag{22}$$

has at least a solution $\boldsymbol{\omega}$ in $D(\mathcal{L})$, for all $\tilde{\boldsymbol{\omega}} \in X$.
By direct calculations, we eliminate the functions $v_i,\ \psi_i$ and α from (22) in order to obtain the following system of equations only in the variables $u_i,\ \varphi_i$ and ϕ

$$\begin{aligned}
&\mathcal{L}_i\, w \equiv \lambda^2 u_i - \frac{1}{\varrho_0}\left[A_{ijmn}\left(u_{n,m} + \varepsilon_{nmk}\varphi_k\right) + B_{ijmn}\varphi_{n,m} + D_{ijk}\phi_{,k}\right]_{,j} = g_i,\\
&\mathcal{L}_{i+3}\, w \equiv \lambda^2 \varphi_i - \frac{1}{I_{ik}}\left[B_{jkmn}\left(u_{n,m} + \varepsilon_{nms}\varphi_s\right) + C_{jkmn}\varphi_{n,m} + E_{jks}\phi_{,s}\right]_{,j} + \\
&\qquad\qquad +\varepsilon_{ijk}\left[A_{jkmn}\left(u_{n,m} + \varepsilon_{nms}\varphi_s\right) + B_{jkmn}\varphi_{n,m} + D_{jkr}\phi_{,r}\right] = g_{i+3}\\
&\mathcal{L}_7\, w \equiv \lambda^2 \sigma - \frac{1}{\varrho_0\kappa}\left[E_{mni}\left(u_{n,m} + \varepsilon_{nms}\varphi_s\right) + E_{mni}\varphi_{n,m} + A_{ij}\phi_{,\,j}\right]_{,\,i} = g_7
\end{aligned} \tag{23}$$

where we used the notations

$$\begin{aligned}
w = (\mathbf{u},\ \boldsymbol{\varphi},\ \phi),\ g_i = \lambda\tilde{u}_i + \tilde{v}_i,\ i = 1,2,3\\
g_{i+3} = \lambda\tilde{\varphi}_i + \tilde{\psi}_i,\ i = 1,2,3,\ \ g_7 = \lambda\tilde{\phi} + \tilde{\alpha}.
\end{aligned} \tag{24}$$

By using the conveniently weighted $[L_2(B)]^7$ inner product $<,>$ we can built the bilinear form $Q\,[w,\ \bar{w}]$ by

$$Q\,[w,\ \bar{w}] = \langle \mathcal{L}\ w,\ \bar{w}\rangle = \left\langle (\mathcal{L}_i\ w,\ \mathcal{L}_{i+3}\ w,\ \mathcal{L}_7\ w)\,,\ \left(\bar{u}_i,\ \bar{\varphi}_i,\ \bar{\phi}\right)\right\rangle =$$
$$= \int_B \left[\varrho_0 \bar{u}_i \mathcal{L}_i\ w + I_{ij}\bar{\varphi}_j \mathcal{L}_{i+3}\ w + \varrho_0 \kappa \bar{\phi} \mathcal{L}_7\ w\right] dV. \tag{25}$$

In this relation we consider (23) and then we use the Green-Gauss formula and the boundary condition (7), so that we are lead to

$$\begin{aligned} Q\,[w,\ \bar{w}] = &\int_B \left[\varrho_0 \lambda^2 u_i u_i + I_{ij}\lambda^2 \varphi_i \varphi_j + \varrho_0 \kappa \lambda^2 \phi^2\right] dV + \\ &+ \int_B [A_{ijmn}\,(u_{n,\ m} + \varepsilon_{nmk}\varphi_k)\,(u_{j,\ i} + \varepsilon_{jis}\varphi_s) + C_{ijmn}\varphi_{n,\ m}\varphi_{j,\ i} + \\ &\quad +2B_{ijmn}\,(u_{j,\ i} + \varepsilon_{jis}\varphi_s)\,\varphi_{n,\ m} + D_{ijk}\,(u_{j,\ i} + \varepsilon_{jis}\varphi_s)\,\phi_{,\ k} + \\ &\qquad +2E_{ijk}\varphi_{j,\ i}\phi_{,\ k} + A_{ij}\phi_{,\ i}\phi_{,\ j}]\, dV, \end{aligned} \tag{26}$$

for any $w = (\mathbf{u},\ \boldsymbol{\varphi},\ \phi) \in \mathbf{H}_0^1(B) \times \mathbf{H}_0^1(B) \times H_0^1(B)$.
Taking into account the hypotheses $i)$ and $ii)$ and first Korn's inequality, we obtain

$$Q\,[w,\bar{w}] \geq C_1 |w|_Y^2,\ for\ \ all\ \ w = (\mathbf{u},\boldsymbol{\varphi},\phi) \in \mathbf{H}_0^1(B) \times \mathbf{H}_0^1(B) \times H_0^1(B), \tag{27}$$

where C_1 is a positive constant, conveniently chosen. Also, if we use the notation $Y \equiv \mathbf{H}_0^1(B) \times \mathbf{H}_0^1(B) \times H_0^1(B)$, then $|w|_Y$ is the norm defined by

$$|w|_Y = |(\mathbf{u},\ \boldsymbol{\varphi},\ \phi)|_Y = |\mathbf{u}|_{\mathbf{H}^1(B)} + |\boldsymbol{\varphi}|_{\mathbf{H}^1(B)} + |\phi|_{H^1(B)}.$$

With the same technique as in the proof of relation (19), we deduce

$$Q\,[w,\ w] \leq C_2 |w|_Y^2,\ \ for\ \ all\ \ w = (\mathbf{u},\ \boldsymbol{\varphi},\ \phi) \in Y. \tag{28}$$

With the help of relations (27) and (28) we deduce that the bilinear form $Q\,[w,\ w]$ determines a norm equivalent to the original norm on the space Y, because we have

$$C_1 |w|_Y^2 \leq Q\,[w,\ w] \leq C_2 |w|_Y^2,\ \ for\ \ all\ \ w = (\mathbf{u},\ \boldsymbol{\varphi},\ \phi) \in Y.$$

Relation (25) ensure that the bilinear form $Q\,[w,\ \bar{w}]$ is continuous on product space $Y \times Y$. So, we obtain that there exists a linear bounded transformation $T : Y \to Y$ such that

$$< w,\ T\bar{w} >_Y = Q\,[w,\ \bar{w}]\,,\ \ for\ \ any\ \ w,\ \bar{w} \in Y. \tag{29}$$

With the help of (25) we deduce

$$< w,\ T\bar{w} >_Y \geq C_1 |w|_Y^2, \tag{30}$$

and then

$$|Tw| \geq C_1 |w|_Y, \quad w \in Y. \tag{31}$$

We choose $\tilde{w} = T\tilde{w}$, such that from (29), it follows that $w \in Y$ is the unique solution of the equation

$$Q\left[w,\ \tilde{w}\right] = \langle \mathbf{g},\ \tilde{w} \rangle,\ \forall\, \tilde{w} \in Y, \tag{32}$$

components of $\mathbf{g}$ being given in (24). Based on relations

$$\lambda u_i - \tilde{u}_i = v_i,\ \lambda \varphi_i - \tilde{\varphi}_i = \psi_i,\ \lambda \phi - \tilde{\phi} = \alpha$$

we deduce that $\mathbf{v} \in \mathbf{H}_0^1(B)$, $\psi \in \mathbf{H}_0^1(B)$ and $\alpha \in H_0^1(B)$.
Now, we have that $\boldsymbol{\omega} = (\mathbf{u},\ \mathbf{v},\ \boldsymbol{\varphi},\ \boldsymbol{\psi},\ \phi,\ \alpha)$ is in $D(\mathcal{L})$ and the proof of Theorem 2 is complete. ■
Now we formulate and prove the main result of our study, namely the continuous dependence of solution of our initial-boundary value problem upon the initial data and supply terms.
To this aim, we denote by $(u_i,\ \varphi_i,\ \phi)$ the difference of two solutions of the problem defined by (9), (7) and (8), corresponding to the difference of the initial data and the difference of body force, body couple and equilibrated extrinsic force, $\boldsymbol{\omega}_0 = \left(\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d},\ \phi^0,\ \phi^1\right)$, $(\mathbf{F},\ \mathbf{G},\ L)$, respectively.

Theorem 3. *Assume that the elastic coefficients are continuously differentiable functions that satisfy the conditions* i) *and* ii). *Moreover, we assume that*

$$\mathbf{F},\ \mathbf{G} \in L_1\left([0,\ t_0]\,; \mathbf{L}_2(B)\right), L \in L_1\left([0,\ t_0]\,; L_2(B)\right)$$

and

$$\mathbf{a} \in \mathbf{H}^1(B),\ \mathbf{b} \in \mathbf{H}^0(B),\ \mathbf{c} \in \mathbf{H}^1(B),\ \mathbf{d} \in \mathbf{H}^0(B),\ \sigma^0 \in H^1(B),\ \sigma^1 \in H^0(B).$$

Then the difference of two solutions of the problem (9), (7) and (8) $(\mathbf{u},\ \boldsymbol{\varphi},\ \phi)$ *satisfies the inequality*

$$\begin{aligned} &|\mathbf{u}|_{\mathbf{H}^1(B)} + |\dot{\mathbf{u}}|_{\mathbf{H}^0(B)} + |\boldsymbol{\varphi}|_{\mathbf{H}^1(B)} + |\dot{\boldsymbol{\varphi}}|_{\mathbf{H}^0(B)} + |\phi|_{H^1(B)} + \left|\dot{\phi}\right|_{H^0(B)} \leq \\ &\leq M \Big\{ |\mathbf{a}|_{\mathbf{H}^1(B)} + |\mathbf{b}|_{\mathbf{H}^0(B)} + |\mathbf{c}|_{\mathbf{H}^1(B)} + |\mathbf{d}|_{\mathbf{H}^0(B)} + \left|\phi^0\right|_{H^1(B)} + \left|\phi^1\right|_{H^0(B)} + \\ &\quad + \int_0^t \left[|\mathbf{F}(\tau)|_{\mathbf{H}^0(B)} + |\mathbf{G}(\tau)|_{\mathbf{H}^0(B)} + |L(\tau)|_{H^0(B)} \right] d\tau \Big\}, \end{aligned} \tag{33}$$

where M is a positive constant.

Proof. The next identity is obtained based on equations (9), the boundary conditions (7) and the initial conditions (8)

$$\int_B \left[\varrho_0 \dot{u}_i \dot{u}_i + I_{ij} \dot{\varphi}_i \dot{\varphi}_j + \varrho_0 \kappa \dot{\sigma}^2\right] dV +$$
$$+ \int_B \left[A_{ijmn} \left(u_{n,m} + \varepsilon_{nmk} \varphi_k\right) \left(u_{j,i} + \varepsilon_{jis} \varphi_s\right) + C_{ijmn} \varphi_{n,m} \varphi_{j,i} + \right.$$
$$+ 2B_{ijmn} \left(u_{j,i} + \varepsilon_{jis} \varphi_s\right) \varphi_{n,m} + 2D_{ijk} \left(u_{j,i} + \varepsilon_{jis} \varphi_s\right) \phi_{,\,k} +$$
$$\left. + 2E_{ijk} \varphi_{j,\,i} \phi_{,\,k} + A_{ij} \phi_{,\,i} \phi_{,\,j}\right] dV =$$

$$= \int_B \left[\varrho_0 \dot{a}_i \dot{a}_i + I_{ij} \dot{c}_i \dot{c}_j + \varrho_0 \kappa \left(\dot{\phi}^0\right)^2\right] dV + \tag{34}$$
$$+ \int_B \left[A_{ijmn} \left(a_{n,m} + \varepsilon_{nmk} c_k\right) \left(a_{j,i} + \varepsilon_{jis} c_s\right) + C_{ijmn} c_{n,m} c_{j,i} + \right.$$
$$+ 2B_{ijmn} \left(a_{j,i} + \varepsilon_{jis} c_s\right) c_{n,m} + 2D_{ijk} \left(a_{j,i} + \varepsilon_{jis} c_s\right) \phi^0_{,\,k} +$$
$$\left. + 2E_{ijk} c_{j,i} \phi^0_{,\,k} + A_{ij} \phi^0_{,\,i} \phi^0_{,\,j}\right] dV +$$
$$+ 2 \int_0^t \int_B \varrho_0 \left[F_i u_i + G_i \varphi_i + L\sigma\right] dV d\tau$$

Finally, we use the Schwarz's inequality, the hypotheses i) and ii) and first Korn's inequality in the identity (34), such that a Gronwall inequality is obtained that proves the desired estimate (33). ∎

Remark. If the boundary conditions (7) are replaced by other boundary conditions, we can prove that the above results are still valid, by using a similar procedure.

Conclusions

In this study we formulate the mixed initial-boundary value problem in the context of the elastodynamic theory of microstretch bodies which is transformed in an abstract temporally evolutionary equation in a Hilbert space, convenient chosen. This allows us to use some results from the theory of semigroups of linear operators to prove the continuous dependence of the solutions upon initial data and supply terms.

References

[1] Adams, R.A.: *Sobolev Spaces*, Academic Press, New York, 1975

[2] Boschi, E. and Mainardi, F.: *Body loadings in thermo-microstretch elastic solids*, Geophys. J. R. Astr. Soc., **34**(1973), 313-320

[3] Ciarletta, M.: *On the bending of microstretch elastic plates*, Int. J. Engng. Sci., **37**(1995), 1309-1318,

[4] Ciarletta, M, Svanadze, M. and Buonanno, L.: *Plane waves and vibrations in the theory of micropolar thermoelasticity for materials with voids*, European J. of Mech., A/Solids, **28**(2009), 897-903,

[5] Eringen, A.C.:*Theory of micromorphic materials with memory*, Int. J. Engng. Sci., **10**(1972), 623-641,

[6] Eringen, A.C.:*Theory of thermo-microstretch elastic solids*, Int. J. Engng. Sci., **28**(1990), 1291-1301,

[7] Hlavacek, I. and Necas, I.: *On inequalities of Korn's type*, Arch. Rational Mech. Anal., **36**(1970), 305-311,

[8] Iesan, D. and Pompei, A: *On the equilibrium theory of microstretch elastic solids*, Int. J. Engng. Sci., **33**(1995), 399-410,

[9] Iesan, D. and Nappa, L.:*Method of complex potentials in linear microstretch elasticity*, Int. J. Engng. Sci., **44** (11-12)(2006), 797-806,

[10] Knops, R.J. and Payne, L.E.: *On uniqueness and continuous dependence in dynamical problems of thermoelasticity*, Int. J. Solids Structures, **6**(1970), 1173-1184,

[11] Vasilescu Anca, Upon the Haskell support for the web applications development, An. t. Univ. Ovidius Constana, Vol. 23(1), 2015, pp. 277-290,

[12] Marin, M.: *The Lagrange identity method in thermoelasticity of bodies with microstructure*, Int. J. Eng. Sci., **32** (8) (1994), 1229-1240,

[13] Marin, M. and Lupu, M.: *On harmonic vibrations in thermoelasticity of micropolar bodies*, Journal of Vibration and Control, vol. **4**(5), 507-518, 1998,

[14] Marin, M. and Marinescu, C.: *Thermoelasticity of initially stressed bodies. Asymptotic equipartition of energies*, Int. J. Eng. Sci., **36** (1) (1998), 73-86,

[15] Marin, M.: *A domain of influence theorem for microstretch elastic materials*, Nonlinear Analysis: Real World Applications, **11** (5) (2010), 3446-3452

[16] Marin, M.: *Some basic theorems in elastostatics of micropolar materials with voids*, J. Comp. Appl. Math., **70**(1) (1996), 115-126,

[17] Pazy, A.: *Semigroups of operators of linear operators and applications*, Springer, New York, Berlin, 1983.

[18] Sharma, K. and Marin, M.: *Reflection and transmission of waves from imperfect boundary between two heat conducting micropolar thermoelastic solids*, An. Sti. Univ. Ovidius Constanta-Math., **22** (2) (2014), 151-175,

[19] Reiss, R.: *Minimum principles for linear elastodynamics*, Journal of Elasticity, **8** (1) (1978), 35-45,

Marin MARIN,
Department of Mathematics and Computer Science,
Transilvania University of Brasov,
Bdul Iuliu Maniu, nr.50, Brasov, Romania.
Email: m.marin@unitbv.ro

Ibrahim ABBAS,
Department of Mathematics, Faculty of Science & Arts-Khulais,
King AbdulAziz University, 21384 Jeddah, Saudi Arabia.
Department of Mathematics,
Faculty of Science, Sohag University, 82524 Sohag, Egypt.
Email: ibrabbas7@gmail.com

Claudia CÂRSTEA,
Department of Mathematics and Computer Science,
Transilvania University of Brasov,
Bdul Iuliu Maniu, nr.50, Brasov, Romania.
Email: carstea.claudia@unitbv.ro